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1997 J. Phys. A: Math. Gen. 30 1831

(http://iopscience.iop.org/0305-4470/30/6/010)

J. Phys. A: Math. Gen. **30** (1997) 1831–1841. Printed in the UK PII: S0305-4470(97)78082-X

Finite-size corrections of an integrable chain with alternating spins

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Received 23 September 1996, in final form 9 December 1996

Abstract. In this paper we calculate the finite-size corrections of an anisotropic integrable spin chain, consisting of spins $s = 1$ and $s = \frac{1}{2}$. The calculations are done in two regions of the phase diagram with respect to the two couplings $\bar{c}$ and $\tilde{c}$. In the case of conformal invariance we obtain the final answer for the ground state and its lowest excitations, which generalizes earlier results.

1. Introduction

In 1992 de Vega and Woynarovich constructed the first example of a spin chain with alternating spins of the values $s = \frac{1}{2}$ and $s = 1$ [1] on the basis of the well known $XXZ(\frac{1}{2})$ model. We call this model $XXZ(\frac{1}{2}, 1)$. Later on many interesting generalizations were presented [2–4]. After de Vega *et al* [5, 6] we studied the $XXZ(\frac{1}{2}, 1)$ model in two subsequent publications [7, 8]. In our last paper [8] we determined the ground state for different values of the two couplings $\bar{c}$ and $\tilde{c}$ (for the details see section 3 of that paper). Disregarding two singular lines we have found four regions in the $(\bar{c}, \tilde{c})$-plane which can be divided into two classes. The division is made with respect to the occurance of finite Fermi zones for Bethe ansatz roots. Only the two regions with infinite Fermi zones have been widely studied [1, 7] in the framework of Bethe ansatz. On that basis we consider the finite-size corrections for the ground state and its lowest excitations using standard techniques [9–11]. It is remarkable that they allow us to obtain an explicit answer only in the conformally invariant cases, which are contained in the two regions considered. The results can easily be compared with the predictions of conformal invariance.

The paper is organized as follows. Definitions are reviewed in section 2. In section 3 we calculate the finite-size corrections for both couplings negative. The same is done in section 4 for positive couplings. Here it was necessary to set $\bar{c} = \tilde{c}$ to obtain explicit answers. Section 5 contains interpretation of the results and our conclusions.

2. Description of the model

We consider the Hamiltonian of a spin chain of length $2N$ with N even:

$$\mathcal{H}(\gamma) = \bar{c}\bar{\mathcal{H}}(\gamma) + \tilde{c}\tilde{\mathcal{H}}(\gamma). \tag{2.1}$$

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The two Hamiltonians can (implicitly) be found in [1], they both contain a two-site and a three-site coupling part. Their explicit expressions are rather lengthy and do not provide any further insights. They include a XXZ-type anisotropy parametrized by $e^{i\gamma}$; we restrict ourselves to $0 < \gamma < \pi/2$. The isotropic limit $XXX(\frac{1}{2}, 1)$ is contained in [2]. The two real coupling constants $\bar{c}$ and $\tilde{c}$ dominate the qualitative behaviour of the model. The interaction favours antiparallel orientation of spins, for equal signs of the couplings its character resembles the ordinary XXZ model. A new kind of competition comes in for different signs of couplings where the ground state is still a singlet, but with a much more involved structure.

The Bethe ansatz equations (BAE) determining the solution of the model are

$$\left(\frac{\sinh(\lambda_j + \frac{1}{2}i\gamma)}{\sinh(\lambda_j - \frac{1}{2}i\gamma)} \frac{\sinh(\lambda_j + i\gamma)}{\sinh(\lambda_j - i\gamma)}\right)^N = -\prod_{k=1}^{M} \frac{\sinh(\lambda_j - \lambda_k + i\gamma)}{\sinh(\lambda_j - \lambda_k - i\gamma)} \qquad j = 1, \ldots, M. \tag{2.2}$$

One can express energy, momentum and spin projection in terms of the BAE roots λ_j:

$$\begin{aligned} E &= \bar{c}\bar{E} + \tilde{c}\tilde{E} \\ \bar{E} &= -\sum_{j=1}^{M} \frac{2\sin\gamma}{\cosh 2\lambda_j - \cos\gamma} \\ \tilde{E} &= -\sum_{j=1}^{M} \frac{2\sin 2\gamma}{\cosh 2\lambda_j - \cos 2\gamma} \end{aligned} \tag{2.3}$$

$$P = \frac{i}{2}\sum_{j=1}^{M}\left\{\ln\left(\frac{\sinh(\lambda_j + \frac{1}{2}i\gamma)}{\sinh(\lambda_j - \frac{1}{2}i\gamma)}\right) + \ln\left(\frac{\sinh(\lambda_j + i\gamma)}{\sinh(\lambda_j - i\gamma)}\right)\right\} \tag{2.4}$$

$$S_z = \tfrac{3}{2}N - M. \tag{2.5}$$

We have defined energy and momentum to vanish for the ferromagnetic state. The momentum operator was chosen to be half of the logarithm of the two-site shift operator [2], which is consistent with taking the length of the system as $2N$ instead of N.

3. Calculation of finite-size corrections for negative couplings

In section 3 of [8] we carried out a detailed analysis of the thermodynamic Bethe ansatz equations (TBAE) at zero temperature and obtained the ground state.

We found a large antiferromagnetic region in the $(\bar{c}, \tilde{c})$-plane (depending on γ) where the ground state is formed by roots with imaginary parts $\frac{1}{2}\pi$, the so-called $(1, -)$ strings. The Fourier transform of their density is given [7] by

$$\hat{\rho}_0(p) = \frac{1 + 2\cosh(p\gamma/2)}{2\cosh(p(\pi - \gamma)/2)}. \tag{3.1}$$

Depending on the signs of $\tilde{c}$ and $\bar{c}$ the region is described by the connection of three parts:

(a) $\qquad \tilde{c} \leqslant 0 \qquad \bar{c} \leqslant 0 \qquad$ (3.2*a*)

(b) $\tilde{c} < 0 \qquad \bar{c} > 0$

$$\frac{\bar{c}}{|\tilde{c}|} \leqslant \frac{1}{2\cos\tilde{\gamma}} \qquad \text{for } 0 < \gamma \leqslant \frac{2\pi}{5} \tag{3.2b}$$

$$\frac{\bar{c}}{|\tilde{c}|} \leqslant 2\cos\tilde{\gamma} \qquad \text{for } \frac{2\pi}{5} \leqslant \gamma < \frac{\pi}{2}$$

(c) $\tilde{c} > 0 \qquad \bar{c} < 0$

$$\frac{|\bar{c}|}{\tilde{c}} \geqslant \frac{8\cos^3\tilde{\gamma}}{4\cos^2\tilde{\gamma} - 1} \qquad \text{for } 0 < \gamma \leqslant \frac{\pi}{3} \tag{3.2c}$$

$$\frac{|\bar{c}|}{\tilde{c}} \geqslant \frac{2}{\cos\tilde{\gamma}} \qquad \text{for } \frac{\pi}{3} \leqslant \gamma < \frac{\pi}{2}.$$

Here for brevity we have introduced

$$\tilde{\gamma} = \frac{\pi\gamma}{2(\pi - \gamma)}. \tag{3.3}$$

We shall now calculate the finite-size corrections for the ground state and its excitations. In [7] the structure of excitations in the framework of the BAE roots was obtained for $\tilde{c} < 0$, $\bar{c} < 0$. Our results immediately apply to the whole region (3.2), because we had to ensure only that the ground state consists of $(1, -)$ strings which follows from the TBAE.

Since we are interested only in the lowest excitations, we disregard the bound states [7] and consider those excitations given by holes in the ground-state distribution which are located to the right (or left) of the real parts of all roots. The number of those holes we call H^+ (H^-). We follow the standard techniques developed in [9, 10].

For transparency we employ the notation of [11] as much as possible. We decompose

$$\sigma_N = \rho_0^{(1)} + \rho_0^{(2)} + \Delta\sigma_N \tag{3.4}$$

where the upper index describes the two terms on the right-hand side of equation (3.1). The basic equations are then

$$\frac{\Delta E_N}{2N} \equiv e_N = \bar{c}\pi \int_{-\infty}^{\infty} \mathrm{d}\lambda\ \rho_0^{(1)}(\lambda) \left\{ \frac{1}{N} \sum_k \delta(\lambda - \lambda_k) - \sigma_N(\lambda) \right\}$$

$$+\tilde{c}\pi \int_{-\infty}^{\infty} \mathrm{d}\lambda\ \rho_0^{(2)}(\lambda) \left\{ \frac{1}{N} \sum_k \delta(\lambda - \lambda_k) - \sigma_N(\lambda) \right\} \tag{3.5}$$

and

$$\Delta\sigma_N(\lambda) = -\int_{-\infty}^{\infty} \mathrm{d}\mu\ \bar{p}(\lambda - \mu) \left\{ \frac{1}{N} \sum_k \delta(\lambda - \lambda_k) - \sigma_N(\lambda) \right\} \tag{3.6}$$

where the Fourier transform of the kernel $\bar{p}(\lambda)$ is given by

$$\bar{P}(\omega) = -\frac{\sinh((\pi/2 - \gamma)\omega)}{2\sinh(\omega\gamma/2)\cosh(\omega(\pi/2 - \gamma/2))}. \tag{3.7}$$

The summation on the right-hand side of (3.5) and (3.6) is carried out over the real parts of all the roots (without the holes). Using the Euler–MacLaurin formula, equations (3.5) and (3.6) are rewritten as usual, e.g.,

$$\sigma_N(\lambda) - \rho_0^{(1)}(\lambda) - \rho_0^{(2)}(\lambda)$$

$$= \int_{\Lambda^+}^{\infty} \mathrm{d}\mu\ \sigma_N(\mu)\bar{p}(\lambda-\mu) - \frac{1}{2N}\bar{p}(\lambda-\Lambda^+) + \frac{1}{12N^2}\frac{1}{\sigma_N(\Lambda^+)}\bar{p}(\lambda-\Lambda^+)$$

$$\left(+ \int_{-\infty}^{\Lambda^-} \mathrm{d}\mu\ \sigma_N(\mu)\bar{p}(\lambda-\mu) - \frac{1}{2N}\bar{p}(\lambda-\Lambda^-)\right.$$

$$\left. - \frac{1}{12N^2}\frac{1}{\sigma_N(\Lambda^-)}\bar{p}(\lambda-\Lambda^-)\right). \tag{3.8}$$

Here Λ^+ (Λ^-) is the real part of the largest (smallest) root. For $\lambda \geqslant \Lambda^+$ the part in round brackets can be omitted and after a shift equation (3.8) converts into a standard Wiener–Hopf problem to be solved. In the expression for ΔE_N both parts have to be kept, so we need the solution for $\lambda \geqslant \Lambda^+$ and $\lambda \leqslant \Lambda^-$, which are simply related (but not equal) by symmetry.

For the solution with $\lambda \geqslant \Lambda^+$, as usual we define

$$X_\pm(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\lambda}\sigma_N^\pm(\lambda+\Lambda^+)\,\mathrm{d}\lambda \tag{3.9}$$

$$\sigma_N^\pm(\lambda+\Lambda^+) = \begin{cases} \sigma_N(\lambda+\Lambda^+) & \text{for } \lambda \gtrless 0 \\ 0 & \text{for } \lambda \lessgtr 0. \end{cases} \tag{3.10}$$

After Fourier transformation equation (3.8) takes the form

$$X_-(\omega) + (1-\bar{P}(\omega))(X_+(\omega) - \bar{C}(\omega)) = \bar{F}_+(\omega) + \bar{F}_-(\omega) - \bar{C}(\omega) \tag{3.11}$$

where we have marked all given functions of our problem by a bar. $\bar{F}_\pm(\omega)$ are defined as above using instead of σ_N the sum $\rho_0^{(1)} + \rho_0^{(2)}$. Moreover

$$\bar{C}(\omega) = \frac{1}{2N} + \frac{\mathrm{i}\omega}{12N^2\sigma_N(\Lambda^+)}. \tag{3.12}$$

Now we have to factorize the kernel

$$[1-\bar{P}(\omega)] = \bar{G}_+(\omega)\bar{G}_-(\omega) \tag{3.13}$$

with $\bar{G}_\pm(\omega)$ holomorphic and continuous in the upper and lower half-planes, respectively. Noting that

$$\bar{P}(\omega,\gamma) = K(\omega, \pi-\gamma) \tag{3.14}$$

where K is the analogous function in [11], we take the factorization from there:

$$\bar{G}_+(\omega) = \sqrt{2\gamma}\,\Gamma\left(1-\frac{\mathrm{i}\omega}{2}\right)\mathrm{e}^{\bar{\psi}(\omega)}\left[\Gamma\left(\frac{1}{2}-\frac{\mathrm{i}(\pi-\gamma)\omega}{2\pi}\right)\Gamma\left(\frac{1}{2}-\frac{\mathrm{i}\gamma\omega}{2\pi}\right)\right]^{-1} = \bar{G}_-(-\omega) \tag{3.15}$$

$$\bar{\psi}(\omega) = \frac{\mathrm{i}\omega}{2}\left[\ln\left(\frac{\pi}{\gamma}\right) - \frac{\pi-\gamma}{\pi}\ln\left(\frac{\pi-\gamma}{\gamma}\right)\right]. \tag{3.16}$$

It is chosen to fulfill

$$\bar{G}_+(\omega) \overset{|\omega|\to\infty}{\sim} 1 + \frac{\bar{g}_1}{\omega} + \frac{\bar{g}_1^2}{2\omega^2} + \mathcal{O}\left(\frac{1}{\omega^3}\right) \tag{3.17}$$

where

$$\bar{g}_1 = \frac{\mathrm{i}}{12}\left(2 + \frac{\pi}{\pi-\gamma} - \frac{2\pi}{\gamma}\right). \tag{3.18}$$

After the necessary decomposition

$$\bar{G}_-(\omega)\bar{F}_+(\omega) = \bar{Q}_+(\omega) + \bar{Q}_-(\omega) \tag{3.19}$$

equation (3.11) has the desired form

$$\frac{X_+(\omega) - \bar{C}(\omega)}{\bar{G}_+(\omega)} - \bar{Q}_+(\omega) = \bar{Q}_-(\omega) - \bar{G}_-(\omega)\left[X_-(\omega) + \bar{C}(\omega) - \bar{F}_-(\omega)\right] \equiv \bar{P}(\omega) \tag{3.20}$$

leading to an entire function $\bar{P}(\omega)$ given by its asymptotics.

$$\bar{P}(\omega) = \frac{\mathrm{i}\bar{g}_1}{12N^2\sigma_N(\Lambda^+)} - \frac{1}{2N} - \frac{\mathrm{i}\omega}{12N^2\sigma_N(\Lambda^+)}. \tag{3.21}$$

Equation (3.20) yields the solution for $X_+(\omega)$:

$$X_+(\omega) = \bar{C}(\omega) + \bar{G}_+(\omega)\left[\bar{P}(\omega) + \bar{Q}_+(\omega)\right]. \tag{3.22}$$

For our purposes it is sufficient to put

$$\bar{F}_+(\omega) = \frac{\exp\big(\pi\Lambda^+/(\pi-\gamma)\big)}{\pi - \mathrm{i}\omega(\pi-\gamma)}(1 + 2\cos\tilde{\gamma}) \tag{3.23}$$

and hence

$$\bar{Q}_+(\omega) = \frac{\bar{G}_+(\mathrm{i}\pi/(\pi-\gamma))\exp\big(-\pi\Lambda^+/(\pi-\gamma)\big)(1+2\cos\tilde{\gamma})}{\pi - \mathrm{i}\omega(\pi-\gamma)}. \tag{3.24}$$

Next we must determine by normalization the value of the integral

$$\int_{\Lambda^+}^{\infty} \sigma_N(\lambda)\,\mathrm{d}\lambda = z_N(\infty) - z_N(\Lambda^+).$$

After a thorough analysis we found for our case that the relation

$$\frac{H}{2} = \frac{H^+ + H^-}{2} = \left[\nu S + \frac{1}{2}\right] \tag{3.25}$$

holds where $\nu = \gamma/\pi$. Nevertheless, we shall not claim that equation (3.25) holds for all possible states [10]; in particluar we expect effects like those described in [12] for higher excitations. Then we have

$$\begin{aligned} z_N(\pm\infty) - z_N(\Lambda^\pm) &= \pm\frac{1}{N}\left(\frac{1}{2} + \nu S_z \pm \frac{H^+ - H^-}{2}\right) \\ &\equiv \pm\frac{1}{N}\left(\frac{1}{2} \pm \Delta^\pm\right) \end{aligned} \tag{3.26}$$

yielding the important equation

$$\begin{aligned} &\frac{\bar{G}_+(\mathrm{i}\pi/(\pi-\gamma))\exp\big(-\pi\Lambda^+/(\pi-\gamma)\big)(1+2\cos\tilde{\gamma})}{\pi} \\ &\quad = \frac{1}{2N} - \frac{\mathrm{i}\bar{g}_1}{12N^2\sigma_N(\Lambda^+)} + \frac{1}{N}\frac{1}{\sqrt{2\nu}}\Delta^+. \end{aligned} \tag{3.27}$$

The other normalization equation is obviously

$$\sigma_N(\Lambda^+) = \frac{\bar{g}_1^2}{24N^2\sigma_N(\Lambda^+)} + \frac{\mathrm{i}\bar{g}_1}{2N} + \frac{\bar{G}_+(\mathrm{i}\pi/(\pi-\gamma))}{\pi-\gamma}\exp\bigl(-\pi\Lambda^+/(\pi-\gamma)\bigr)(1+2\cos\tilde{\gamma}). \tag{3.28}$$

Now we can proceed in the usual way, keeping in mind the changes arising especially from the last two equations:

$$\frac{\Delta E_N}{2N} = -\frac{\pi}{\pi-\gamma}(\bar{c}+2\tilde{c}\cos\tilde{\gamma})\bar{G}_+\left(\frac{\mathrm{i}\pi}{\pi-\gamma}\right)\left[\bar{P}\left(\frac{\mathrm{i}\pi}{\pi-\gamma}\right)\right.$$
$$\left. + \bar{Q}_+\left(\frac{\mathrm{i}\pi}{\pi-\gamma}\right)\right]\exp\bigl(-\pi\Lambda^+/(\pi-\gamma)\bigr) + \bigl(\Lambda^+ \leftrightarrow \Lambda^-\bigr). \tag{3.29}$$

After some algebra using equations (3.27) and (3.28), this turns out as

$$\frac{\Delta E_N}{2N} = -\frac{\pi^2}{\pi-\gamma}\frac{(\bar{c}+2\tilde{c}\cos\tilde{\gamma})}{1+2\cos\tilde{\gamma}}\left\{\left(-\frac{1}{24N^2}+\frac{1}{4N^2\nu}(\Delta^+)^2\right)\right.$$
$$\left. + \left(-\frac{1}{24N^2}+\frac{1}{4N^2\nu}(\Delta^-)^2\right)\right\}. \tag{3.30}$$

Finally, for further interpretation we put it in the form

$$\frac{\Delta E_N}{2N} = -\frac{2\bar{c}+4\tilde{c}\cos\tilde{\gamma}}{1+2\cos\tilde{\gamma}}\frac{\pi}{\pi-\gamma}\left\{-\frac{\pi}{6}\frac{1}{4N^2}+\frac{2\pi}{4N^2}\left(\frac{S_z^2\nu}{2}+\frac{\Delta^2}{2\nu}\right)\right\} \tag{3.31}$$

with $\Delta = (H^+ - H^-)/2$ as an integer number.

The momentum correction ΔP_N is obtained from relation (3.5) after substituting the hole energy $\varepsilon_h = -2\bar{c}\pi\rho_0^{(1)} - 2\tilde{c}\pi\rho_0^{(2)}$ by the hole momentum (see [8])

$$p_h(\lambda) = \tfrac{1}{2}\arctan(\sinh(\pi\lambda)/(\pi-\gamma)) + \arctan(\sinh(\pi\lambda)/(\pi-\gamma))/\cos\tilde{\gamma}) + \text{constant}.$$

Comparing the asymptotics for large λ of both $\varepsilon_h(\lambda)$ and $p_h(\lambda)$ gives the speed of sound and helps to shorten the calculation of ΔP_N. Therefore

$$\frac{\Delta P_N}{2N} = \frac{\pi}{2}\left\{\frac{1}{4N^2\nu}\left[(\Delta^-)^2-(\Delta^+)^2\right]\right\} + \text{constant}. \tag{3.32}$$

We are not interested in the constant term, it being some multiple of π.

Finally

$$\Delta P_N = -\frac{2\pi}{2N}S_z\Delta. \tag{3.33}$$

The interpretation of our result will be given in section 5. We stress once more that to obtain equations (3.31) and (3.33) it was not necessary to put $\bar{c} = \tilde{c}$. The coupling constants are only constrained to stay in the region (3.2).

4. Calculation of finite-size corrections for positive couplings

Now we consider region $\bar{c} > 0$, $\tilde{c} > 0$ and rely on the analysis of [1].

The ground state is given by two densities $\sigma_N^{(1/2)}(\lambda)$ for the real roots and $\sigma_N^{(1)}(\lambda)$ for the real parts of the $(2,+)$ strings. One has

$$\sigma_\infty^{(1/2)}(\lambda) = \sigma_\infty^{(1)}(\lambda) = \frac{1}{2\gamma\cosh(\pi\lambda/\gamma)} \equiv s(\lambda). \tag{4.1}$$

The physical excitations are holes in those distributions. As in section 3 we consider only holes situated to the right (or left) of all roots. With the usual technique and the results of [1] we have obtained after some lengthy but straightforward calculations the basic system for the density corrections

$$\Delta\sigma_N^{(1/2)}(\lambda) = -\int_{-\infty}^{\infty} \mathrm{d}\mu\ \ s(\lambda-\mu)\left\{\frac{1}{N}\sum_{j=1}^{M_1}\delta(\mu-\xi_j) - \sigma_N^{(1)}(\mu)\right\}$$

$$\begin{aligned}\Delta\sigma_N^{(1)}(\lambda) = &-\int_{-\infty}^{\infty} \mathrm{d}\mu\, s(\lambda-\mu)\left\{\frac{1}{N}\sum_{i=1}^{M_{1/2}}\delta(\mu-\lambda_i) - \sigma_N^{(1/2)}(\mu)\right\} \\ &-\int_{-\infty}^{\infty} \mathrm{d}\mu\, r(\lambda-\mu)\left\{\frac{1}{N}\sum_{j=1}^{M_1}\delta(\mu-\xi_j) - \sigma_N^{(1)}(\mu)\right\}.\end{aligned} \tag{4.2}$$

We have denoted the real roots by λ_i (their number is $M_{1/2}$) and the real parts of the strings by ξ_j (their number is M_1). The function $r(\lambda)$ is given via its Fourier transform

$$R(\omega) = \frac{\sinh(\omega(\pi-3\gamma)/2)}{2\sinh(\omega(\pi-2\gamma)/2)\cosh(\omega\gamma/2)}. \tag{4.3}$$

The energy correction takes the form

$$\begin{aligned}\frac{\Delta E_N}{2N} = &-\pi\bar{c}\int_{-\infty}^{\infty} \mathrm{d}\lambda\ s(\lambda)\left\{\frac{1}{N}\sum_{i=1}^{M_{1/2}}\delta(\lambda-\lambda_i) - \sigma_N^{(1/2)}(\lambda)\right\} \\ &-\pi\tilde{c}\int_{-\infty}^{\infty} \mathrm{d}\lambda\ s(\lambda)\left\{\frac{1}{N}\sum_{j=1}^{M_1}\delta(\lambda-\xi_j) - \sigma_N^{(1)}(\lambda)\right\}.\end{aligned} \tag{4.4}$$

Once again we shall follow [11]. The maximum (minimum) real roots we call $\Lambda_{1/2}^{\pm}$ and for the strings we use $\Lambda_1^{\pm}$, respectively. Instead of one $C(\omega)$ we now have $C_1(\omega)$ and $C_{1/2}(\omega)$ generalized in an obvious way. The same applies to $F(\omega)$. The main mathematical problem is the factorization of a matrix kernel

$$(1-K(\omega))^{-1} = G_+(\omega)G_-(\omega) \qquad \text{with} \qquad G_-(\omega) = G_+(-\omega)^{\mathrm{T}} \tag{4.5}$$

(see [5]) and

$$K(\omega) = \begin{pmatrix} 0 & S(\omega)\exp\big(-\mathrm{i}\omega(\Lambda_1^+ - \Lambda_{1/2}^+)\big) \\ S(\omega)\exp\big(\mathrm{i}\omega(\Lambda_1^+ - \Lambda_{1/2}^+)\big) & R(\omega) \end{pmatrix}. \tag{4.6}$$

G_+ is now a matrix function and G_+^{T} stands for its transposition. The two-component vector $Q_+(\omega)$ (see equation (3.19)) is

$$Q_+(\omega) = \frac{G_+(\mathrm{i}\pi/\gamma)^{\mathrm{T}}}{\pi - \mathrm{i}\omega\gamma}\begin{pmatrix}\exp\big(-\pi\Lambda_{1/2}^+/\gamma\big) \\ \exp\big(-\pi\Lambda_1^+/\gamma\big)\end{pmatrix}. \tag{4.7}$$

As usual we define the constant matrices G_1 and G_2 by

$$G_+(\omega) \overset{|\omega|\to\infty}{\longrightarrow} 1 + G_1\frac{1}{\omega} + G_2\frac{1}{\omega^2} + \mathcal{O}\left(\frac{1}{\omega^3}\right) \tag{4.8}$$

and as before one has

$$G_2 = \tfrac{1}{2}G_1^2. \tag{4.9}$$

The two-component vector $P(\omega)$ is then

$$P(\omega) = \begin{pmatrix} -\frac{1}{2N} - \frac{\mathrm{i}\omega}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ -\frac{1}{2N} - \frac{\mathrm{i}\omega}{12N^2\sigma_N^{(1)}(\Lambda_1^+)} \end{pmatrix} + G_1 \begin{pmatrix} \frac{\mathrm{i}}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ \frac{\mathrm{i}}{12N^2\sigma_N^{(1)}(\Lambda_1^+)} \end{pmatrix} \tag{4.10}$$

and therefore the shifted densities are expressed in the form

$$\begin{pmatrix} X_{1/2}^+(\omega) \\ X_1^+(\omega) \end{pmatrix} = \begin{pmatrix} C_{1/2}(\omega) \\ C_1(\omega) \end{pmatrix} + G_+(\omega)\left[P(\omega) + Q_+(\omega)\right]. \tag{4.11}$$

Now it is necessary to find the analogue of equation (3.26) for the two counting functions. Here it would be necessary to consider different cases depending on the fractions of $\nu S_z/N$ or $2\nu S_z/N$. From our experience we know that the result of the finite-size corrections does not depend on those fractions, while relations like (3.25) obviously do. As we are interested only in the former, we shall proceed as straightforwardly as possible and consider only the case with vanishing: fractions.

$$z_N^{(1/2)}(\pm\infty) - z_N^{(1/2)}(\Lambda_{1/2}^\pm) = \pm\frac{1}{N}\left(\frac{1}{2} - \nu S_z + H_{1/2}^\pm\right) \tag{4.12}$$

$$z_N^{(1)}(\pm\infty) - z_N^{(1)}(\Lambda_1^\pm) = \pm\frac{1}{N}\left(\frac{1}{2} - 2\nu S_z + H_1^\pm\right). \tag{4.13}$$

Easy counting leads to expressions for the numbers of the holes:

$$\begin{aligned} H_1 &= 2S_z \\ H_{1/2} &= 2S_z + 2M_1 - N. \end{aligned} \tag{4.14}$$

We expect modifications to these for non-vanishing fractions. We stress that both numbers are even.

Equation (3.27) is now more complicated:

$$\frac{G_+(\mathrm{i}\pi/\gamma)^{\mathrm{T}}}{\pi}\begin{pmatrix} \exp\left(-\pi\Lambda_{1/2}^+/\gamma\right) \\ \exp\left(-\pi\Lambda_1^+/\gamma\right) \end{pmatrix} = G_+^{-1}(0)B^+ + \begin{pmatrix} \frac{1}{2N} \\ \frac{1}{2N} \end{pmatrix} - \mathrm{i}G_1\begin{pmatrix} \frac{1}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ \frac{1}{12N^2\sigma_N^{(1)}(\Lambda_1^+)} \end{pmatrix} \tag{4.15}$$

with the definitions

$$B^\pm = \begin{pmatrix} B_1^\pm \\ B_2^\pm \end{pmatrix} = \frac{1}{N}\begin{pmatrix} -\nu S_z + H_{1/2}^\pm \\ -2\nu S_z + H_1^\pm \end{pmatrix} = \frac{1}{N}\begin{pmatrix} S_z - \nu S_z + M_1 - \frac{1}{2}N \pm \Delta^{(1/2)} \\ S_z - 2\nu S_z \pm \Delta^{(1)} \end{pmatrix} \tag{4.16}$$

and

$$\Delta^{(i)} = \frac{H_i^+ - H_i^-}{2}. \tag{4.17}$$

The other normalization condition is obviously

$$\begin{aligned} \begin{pmatrix} \sigma_N^{(1/2)}(\Lambda_{1/2}^+) \\ \sigma_N^{(1)}(\Lambda_1^+) \end{pmatrix} &= \frac{G_1^2}{2}\begin{pmatrix} \frac{1}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ \frac{1}{12N^2\sigma_N^{(1)}(\Lambda_1^+)} \end{pmatrix} + \mathrm{i}G_1\begin{pmatrix} \frac{1}{2N} \\ \frac{1}{2N} \end{pmatrix} \\ &\quad + \frac{G_+(\mathrm{i}\pi/\gamma)^{\mathrm{T}}}{\gamma}\begin{pmatrix} \exp\left(-\pi\Lambda_{1/2}^+/\gamma\right) \\ \exp\left(-\pi\Lambda_1^+/\gamma\right) \end{pmatrix}. \end{aligned} \tag{4.18}$$

After combining equations (4.18) and (4.15), from equation (4.4) we obtain

$$\frac{\Delta E_N}{2N} = \frac{\pi}{\gamma}\begin{pmatrix}\bar{c}\exp\big(\pi\Lambda^+_{1/2}/\gamma\big)\\ \tilde{c}\exp\big(\pi\Lambda^+_1/\gamma\big)\end{pmatrix}^{\mathrm{T}} G_+\left(\frac{\mathrm{i}\pi}{\gamma}\right)\left[-\frac{1}{2}\begin{pmatrix}1/2N\\1/2N\end{pmatrix}\right.$$
$$\left. + \left(\frac{1}{2}\mathrm{i}G_1 + \frac{\pi}{\gamma}\right)\begin{pmatrix}1/12N^2\sigma_N^{(1/2)}(\Lambda^+_{1/2})\\ 1/12N^2\sigma_N^{(1)}(\Lambda^+_1)\end{pmatrix} + \frac{1}{2}G_+^{-1}(0)B^+\right]$$
$$+ (\Lambda^+ \leftrightarrow \Lambda^-, B^+ \leftrightarrow B^-). \tag{4.19}$$

This result is valid for any positive $\bar{c}$ and $\tilde{c}$. As in [5] no further progress can be made unless the factorization is explicitly known, which up to now has not been the case. For $\bar{c} = \tilde{c} = c$ (conformal invariance) the problem simplifies and the final answer can be obtained.

After some lengthy calculations one arrives at

$$\frac{\Delta E_N}{2N} = \frac{\pi^2 c}{\gamma}\left[-\frac{1}{12N^2} + \frac{1}{2}\left(\big(B_1^+ - B_2^+\big)^2 + B_1^+B_2^+ - (B_2^+)^2\frac{\pi - 3\gamma}{2(\pi - 2\gamma)}\right)\right] + (B^+ \leftrightarrow B^-) \tag{4.20}$$

which can be simplified to give

$$\frac{\Delta E_N}{2N} = \frac{2\pi c}{\gamma}\left\{-\frac{\pi}{6}\frac{2}{4N^2} + \frac{2\pi}{4N^2}\left[\frac{1}{4}(1-2\nu)S_z^2 + \frac{1}{4}\left(H_{1/2} - \frac{H_1}{2}\right)^2\right.\right.$$
$$\left.\left. + \ (\Delta^{(1/2)})^2 - \Delta^{(1/2)}\Delta^{(1)} + \frac{1}{2}(\Delta^{(1)})^2\frac{1-\nu}{1-2\nu}\right]\right\}. \tag{4.21}$$

For $\bar{c} \neq \tilde{c}$ we expect the result to be much more complicated, but of the same order $1/N^2$.

As above the momentum correction is given from relation (4.4) after replacing $\varepsilon_h^{(1/2)} = 2\bar{c}\pi s(\lambda)$ by $p_h^{(1/2)} = \arctan \mathrm{e}^{\pi\lambda/\gamma}$ and $\varepsilon_h^{(1)} = 2\tilde{c}\pi s(\lambda)$ by $p_h^{(1)} = \arctan \mathrm{e}^{\pi\lambda/\gamma}$. The values of the hole momenta are taken from [1] where an additionel factor $\frac{1}{2}$ must be introduced to take into account our definition of momentum (2.4).

As in section 3, comparing the asymptotics for $\bar{c} = \tilde{c}$ gives the speed of sound and, together with equation (4.20), the momentum correction

$$\frac{\Delta P_N}{2N} = \frac{\pi}{2}\left\{\frac{1}{2}\left[\big(B_1^- - B_2^-\big)^2 - \big(B_1^+ - B_2^+\big)^2\right.\right.$$
$$\left.\left. + \ B_1^-B_2^- - B_1^+B_2^+ - ((B_1^-)^2 - (B_1^+)^2)\frac{\pi - 3\gamma}{2(\pi - 2\gamma)}\right]\right\} + \text{constant}. \tag{4.22}$$

Disregarding the constant (multiple of π) we have

$$\Delta P_N = \frac{\pi}{2}\left\{-\Delta^{(1/2)}\left(H_{1/2} - \frac{H_1}{2}\right) - \Delta^{(1)}\left(\frac{H_1}{2} - \frac{H_{1/2}}{2}\right)\right\}. \tag{4.23}$$

5. Conclusions

In sections 3 and 4 we have determined the finite-size corrections of our model for two different cases. Equations (3.31) and (3.33) give the result for region (3.2), while equations (4.21) and (4.23) are valid for $\bar{c} = \tilde{c} = c > 0$. In both cases we have conformal invariance. That seems to be the reason why during the calculations many terms cancel each other and the result is simplified considerably.

From the asymptotics of ε_h and p_h we find

$$v_{\mathrm{s}} = -\frac{2\pi}{\pi-\gamma}\frac{\bar{c}+2\tilde{c}\cos\tilde{\gamma}}{1+2\cos\tilde{\gamma}} > 0 \tag{5.1}$$

for the speed of sound and from relation (3.31) the value of 1 for the central charge. Equation (5.1) generalizes the former result obtained for equal (but negative) couplings [8]. Analogously it follows that

$$v_{\mathrm{s}} = \frac{2c\pi}{\gamma} \tag{5.2}$$

and the central charge equals 2 [1].

For completeness, we mention the heat capacities per site at low temperature given by

$$C = \frac{c_{\mathrm{v}}T\pi}{3v_{\mathrm{s}}} \tag{5.3}$$

where we have denoted the central charge by c_{v} to avoid confusion. Therefore

$$C = -\frac{1+2\cos\tilde{\gamma}}{\bar{c}+2\tilde{c}\cos\tilde{\gamma}}\frac{(\pi-\gamma)T}{6} \tag{5.4}$$

and

$$C = \frac{\gamma T}{3c} \tag{5.5}$$

in agreement with former results [8]. Equation (5.4) generalizes our calculations for $\bar{c} = \tilde{c}$.

The dimensions x_n and the spins s_n of the primary operators follow from equations (3.31) and (3.33):

$$x_n = \frac{S_z^2\nu}{2} + \frac{\Delta^2}{2\nu} \tag{5.6}$$

$$s_n = S_z|\Delta| \tag{5.7}$$

for negative coupling. It is interesting to compare this with the result for the $XXZ(\frac{1}{2})$ model, where ν is simply replaced by $1-\nu$. Equations (5.6) and (5.7) have to be understood in the way that for general excited states (arbitrary holes and complex roots) S_z and Δ are replaced by more complicated integer numbers [10].

For positive couplings

$$x_n = \frac{1}{4}(1-2\nu)S_z^2 + \frac{1}{4}\left(H_{1/2} - \frac{H_1}{2}\right)^2 + \left(\Delta^{(1/2)}\right)^2 - \Delta^{(1/2)}\Delta^{(1)} + \frac{1}{2}\left(\Delta^{(1)}\right)^2\frac{1-\nu}{1-2\nu}. \tag{5.8}$$

When $\nu \to 0$ the first two terms agree with [4], while the other terms are connected with the asymmetry of the state which was not considered there. The dimension of a general primary operator depends on four integer numbers. The second of them measures an asymmetry between the number of holes among real roots or strings, respectively. Once more, for more complicated states the integers in equation (5.8) are replaced by other ones depending on the concrete structure of the state. We mention that relation (5.8) can be 'diagonalized' to resemble the expression of two models both of central charge 1:

$$x_n = \frac{1}{2}\frac{(1-2\nu)}{2}S_z^2 + \frac{1}{2}\frac{\left(H_{1/2}-H_1/2\right)^2}{2} + \frac{1}{2}2\Delta_1^2 + \frac{1}{2}\frac{2}{1-2\nu}\Delta_2^2 \tag{5.9}$$

with

$$\begin{aligned} \Delta_1 &= \Delta^{(1/2)} - \tfrac{1}{2}\Delta^{(1)} \\ \Delta_2 &= \tfrac{1}{2}\Delta^{(1)}. \end{aligned} \tag{5.10}$$

Expression (5.9) becomes even more symmetric if one remembers the first equation of (4.14) and the definition (4.17). Then twice a certain number of holes is linked with its appropriate asymmetry.

From equation (4.23) the spins of the primary operators are

$$s_n = \left|\Delta^{(1/2)}\left(H_{1/2} - \frac{H_1}{2}\right) + \Delta^{(1)}\left(\frac{H_1}{2} - \frac{H_{1/2}}{2}\right)\right| \tag{5.11}$$

and after using (5.10)

$$s_n = \left|\Delta_1\left(H_{1/2} - \frac{H_1}{2}\right) + \Delta_2 S_z\right| \tag{5.12}$$

with the same symmetry as relation (5.9).

Finally, we determine the magnetic susceptibilities per site at zero temperature and vanishing field from our finite-size results. That can be done, because the states we have considered include those with minimum energy for a given S_z (magnetization) [13]. Differentiating twice the energy with respect to S_z gives the inverse susceptibility.

Hence

$$\chi = \frac{\pi - \gamma}{4\pi\gamma}\left(-\frac{1 + 2\cos\tilde{\gamma}}{\bar{c} + 2\tilde{c}\cos\tilde{\gamma}}\right) = \frac{1}{v_s}\frac{1}{2\gamma} \tag{5.13}$$

and

$$\chi = \frac{1}{2c\pi}\frac{1}{\pi - 2\gamma} = \frac{1}{v_s}\frac{1}{\pi - 2\gamma} \tag{5.14}$$

respectively for the two cases, in agreement with earlier results [4, 5, 8]. Expression (5.13) for general couplings in the region (3.2) had not been derived before.

Acknowledgment

One of us (StM) would like to thank H J de Vega for helpful discussions.

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